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A googly amplitude from the B-model in twistor space

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Abstract: Recently it has been proposed that gluon scattering amplitudes in gauge theory can be computed from the D-instanton expansion of the topological B-model on $\mathbb{P}^{3|4}$, although only maximally helicity violating (MHV) amplitudes have so far been obtained from a direct B-model calculation. In this note we compute the simplest non-MHV gluon amplitudes $(++---$ and $+-+--)$ from the B-model as an integral over the moduli space of degree 2 curves in $\mathbb{P}^{3|4}$ and find perfect agreement with Yang-Mills theory.

Keywords: String Duality, QCD.

Contents

1. Introduction

The relation between gauge theory and string theory is one of the most important themes in modern theoretical physics. Significant progress in understanding this relation has emerged from the AdS/CFT correspondence, but this formulation does not allow direct access to the perturbative states of the gauge theory, making it difficult to calculate experimentally relevant Yang-Mills (YM) scattering amplitudes.

Recently, Witten [1] described a remarkable construction which formulates $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory as a full-fledged string theory, the B-twisted topological string on $\mathbb{P}^{3|4}$. It was conjectured that the perturbative amplitudes of YM theory are recovered from instantonic D1-string calculations in this topological string theory. The origin of this proposal was the observation that YM scattering amplitudes have unexpected properties which seem to cry out for some deeper explanation. For example, it has been known since the work of Nair [2] that maximally helicity violating (MHV) tree-level amplitudes can be written in terms of correlation functions of free fermionic currents on a 2-sphere. In [1] it was shown, by considering a large number of examples, that more complicated YM amplitudes satisfy a number of highly nontrivial differential identities. These identities express the fact that YM scattering amplitudes, when transformed to the twistor space [3] $\mathbb{P}^3$ of Minkowski space, are supported on curves whose genus and degree are related to the number of YM loops and to the number and helicities of the external legs. The topological B-model was proposed as a candidate theory which would expose these remarkable properties.

This proposal was used in [1] to provide a context for the calculation of [2] and to reproduce the MHV gluon scattering amplitudes from the topological B-model by integrating a certain free-fermion correlation function over the moduli space of degree one curves in $\mathbb{P}^{3|4}$. However, the question of whether the conjecture might prove computationally useful for more complicated amplitudes was left open.

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On the gauge theory side there exists a wealth of results on YM scattering amplitudes, derived through a variety of methods. The complexity of the result grows substantially as additional negative helicity gluons are added. Tree level MHV amplitudes (and their conjugates) with an arbitrary number of external legs have been computed in [4, 5]. Powerful recurrence relations constructed in [6] were exploited in [7] to calculate certain amplitudes with three negative helicity and arbitrarily many positive helicity gluons. General algorithms simplifying the calculations for theories with massless particles have been devised in [8]. Substantial progress was achieved also in the calculation of loop amplitudes by the use of string-inspired methods [9] and the exploitation of the constraints coming from collinear limits and unitarity [10, 11].

In this note we provide strong further evidence for the conjecture of [1] by recovering the gauge theory 5-point amplitude with three negative and two positive helicity[1] gluons $(++---$ and $+-+--)$ from a D-instanton computation in the open string field theory of the B-model on $\mathbb{P}^{3|4}$. These amplitudes are quite simple in gauge theory, since in lorentzian signature they are complex conjugates of MHV amplitudes. However the B-model calculation involves an apparently quite nontrivial integral over the moduli space of degree 2 curves in $\mathbb{P}^{3|4}$.

It may appear that our calculation amounts to using an elephant gun to shoot a fly. We are optimistic however that the calculational techniques employed in this paper will generalize, with some refinement, to more complicated amplitudes. In particular, since we now know that it is possible to evaluate an integral over the moduli space of degree 2 curves, it does not appear exceedingly difficult to add an arbitrary number of additional positive helicity gluons (which do not change the degree). The corresponding gauge theory amplitudes, in the few cases which are known, are rather complicated [7]. Ultimately the correspondence between YM theory and the topological B-model may provide powerful new insights as well as concrete calculational tools for studying gauge theory amplitudes.

We begin in section 2 with a review of the gauge theory results and of the observation that they are supported on certain classes of curves in the twistor space of Minkowski space and proceed in section 3 to briefly review the string theory construction. We then use the general expression for the scattering amplitudes described in that section to recover in section 4 the gauge theory result for the 5-point amplitude.

2. Helicity amplitudes in gauge theory

We consider tree-level scattering amplitudes $\mathcal{A}_n$ of n gluons in YM theory. At tree level, the amplitudes do not depend on the presence or absence of supersymmetry. All formulae will be written in a manifestly $\mathcal{N}=4$ supersymmetric way, but the resulting gluon amplitudes are equally valid in theories with less supersymmetry, such as QCD.

The most compact expressions for these amplitudes are obtained with the help of two very efficient notational devices: color ordering and the spinor helicity notation (see for

[1] Amplitudes with two positive helicity gluons and an arbitrary number of negative helicity ones were called ‘googly’ MHV amplitudes in [1].

example [12] for a review). Color ordering means that we write the total n-gluon amplitude as a sum over non-cyclic permutations ρ of the n external legs

$$\mathcal{A}_n = \sum_{\rho} \mathrm{Tr}(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}})\, \widehat{A}(\rho(1), \ldots, \rho(n))\,, \tag{2.1}$$

where T^a are generators of the gauge group in the adjoint representation. The color-ordered amplitude $\widehat{A}$ is invariant under cyclic permutations of the external legs and has all of the gauge group structure stripped away. We can therefore study $\widehat{A}$ without needing to specify any particular gauge group.

The spinor helicity notation relies on the fact that any null vector p_μ can be decomposed as

$$p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}} \tag{2.2}$$

into a pair of (commuting) spinors of opposite chirality.[2] Furthermore, for any chosen pair $(\lambda_a, \widetilde{\lambda}_{\dot{a}})$, it is possible to construct polarization vectors $\epsilon^\mu_\pm$ of either positive or negative helicity, which are each unique up to gauge transformations. We use the epsilon tensor to raise and lower the a and $\dot{a}$ indices, and we introduce the inner products

$$\langle i, j\rangle \equiv \langle \lambda_i, \lambda_j\rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b\,, \qquad [i, j] \equiv [\widetilde{\lambda}_i, \widetilde{\lambda}_j] = \epsilon_{\dot{a}\dot{b}} \widetilde{\lambda}_i^{\dot{a}} \widetilde{\lambda}_i^{\dot{b}}\,. \tag{2.3}$$

Scattering amplitudes are conveniently expressed not as a function $\widehat{A}(p_\mu, \epsilon_\mu)$ of the momenta and polarizations of the n particles, but rather as a function $\widehat{A}(\lambda_a, \widetilde{\lambda}_{\dot{a}})$ of the spinors, with the particle helicities specified.

Amplitudes in which all or all but one of the n gluons have the same helicity vanish. The first non-trivial case, in which $n-2$ gluons have positive helicity and two gluons have negative helicity, is called the maximally helicity violating (MHV) amplitude. The MHV amplitude for n gluons in QCD is given by the Parke-Taylor formula [4, 5], whose $\mathcal{N} = 4$ supersymmetric generalization may be written as [2]

$$\widehat{A}_{\mathrm{MHV}}(\lambda, \widetilde{\lambda}, \eta) = i g^{n-2} (2\pi)^4 \delta^4 \left(\sum_{i=1}^n \widetilde{\lambda}_i^{\dot{a}} \lambda_i^a \right) \delta^8 \left(\sum_{i=1}^n \lambda_i^a \eta_{iA} \right) \prod_{i=1}^n \frac{1}{\langle i, i+1\rangle}\,, \tag{2.4}$$

where η_A, $A = 1, 2, 3, 4$ are superspace coordinates and $n + 1 \cong 1$ is understood.

An amplitude $\widehat{A}(\lambda, \widetilde{\lambda}, \eta)$ written in physical space coordinates can be expressed in twistor space variables by Fourier transform:

$$\widetilde{A}(\lambda, \mu, \psi) = \int \frac{d^{2n}\widetilde{\lambda}}{(2\pi)^{2n}}\, d^{4n}\eta\, \exp\left[i \sum_{i=1}^n \left([\mu_i, \widetilde{\lambda}_i] + \psi_i^A \eta_{iA} \right) \right] \widehat{A}(\lambda, \widetilde{\lambda}, \eta)\,. \tag{2.5}$$

As reviewed in [1], YM scattering amplitudes are always homogeneous of degree zero in the variables λ, μ, and ψ, so an amplitude $\widetilde{A}(\lambda, \mu, \psi)$ may be viewed as a function not on

[2] In lorentzian signature, $\widetilde{\lambda}$ and λ are related by complex conjugation. In signature $--++$, λ and $\widetilde{\lambda}$ may be chosen to be independent real variables, and we will do so in this paper since this signature is the one for which the connection to the string theory on twistor space is most straightforward. On the YM side this unusual signature introduces no real difficulty since tree-level amplitudes are easily continued to lorentzian signature. At loop level the situation is less clear.

$\mathbb{C}^{4|4}$ but on $\mathbb{P}^{3|4}$, which is the (super-) twistor space of (super-) Minkowski space. This space has homogeneous coordinates (z^I, ψ^A), $I = 0, \ldots, 3$, $A = 1, \ldots, 4$ which are identified according to

$$(z^I, \psi^A) \cong (tz^I, t\psi^A) \tag{2.6}$$

for any non-zero complex number t. For the present application we decompose the bosonic coordinates into $z^I = (\lambda^1, \lambda^2, \mu^1, \mu^2)$.

In [1] it was conjectured that in twistor space, the n-particle scattering amplitudes with q negative helicity and $n-q$ positive helicity gluons are supported on curves in $\mathbb{P}^{3|4}$ of degree

$$d = q - 1 + l\,, \tag{2.7}$$

and genus

$$g \leq l \tag{2.8}$$

where l is the number of YM loops.

For the MHV amplitudes at tree level, it is easily shown by evaluating the Fourier transform of (2.4) that the amplitude is supported on curves of degree 1 in twistor space [1]. For higher degree the Fourier transform appears very complicated. Fortunately, we will see in section 4 that it is much simpler to take the Fourier transform directly in the topological B-model before evaluating the more complicated integral over instanton moduli space.

The simplest non-MHV amplitudes are those with five gluons of helicities $(++---)$ or $(+-+--)$. In lorentzian signature, amplitudes with $n-2$ negative helicities and 2 positive helicities are related to MHV amplitudes by complex conjugation and therefore are given by the simple formula

$$\begin{aligned}\widehat{A}_{\overline{\mathrm{MHV}}}(\lambda, \widetilde{\lambda}, \eta) &= ig^{n-2}(2\pi)^4\delta^4\left(\sum_{i=1}^{n}\widetilde{\lambda}_i^{\dot{a}}\lambda_i^a\right)\times \\ &\times\int d^{4n}\psi\,\exp\left[i\sum_{i=1}^{n}\eta_{iA}\psi_i^A\right]\delta^8\left(\sum_{i=1}^{n}\widetilde{\lambda}_i^{a}\psi_i^A\right)\prod_{i=1}^{n}\frac{1}{[i, i+1]}\,.\end{aligned} \tag{2.9}$$

In section 4 we recover the precise formula (2.9), for $n = 5$, from a particular amplitude in the presence of a D1-instanton of degree 2 in the topological B-model.

3. How to calculate amplitudes in the B-model on $\mathbb{P}^{3|4}$

A string theory describing $\mathcal{N} = 4$ SYM theory without additional states must clearly be different from the usual critical string theories which contain infinite towers of massive string states. A few obvious constraints are: the theory should have a finite spectrum, it should be globally invariant under the four dimensional superconformal group SU(2, 2|4), its target space should be related to the usual four dimensional space without the introduction of additional compact dimensions and, of course, it should reproduce the scattering amplitudes of the gauge theory.

The proposal put forward in [1] is that the string field theory (SFT) of the open topological B-model whose target space is the supermanifold $\mathbb{P}^{3|4}$ with D5 and D1 branes describes $\mathcal{N} = 4$ SYM.

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It is clear that this theory has the first three properties stated above. In the presence of N D5 branes and no D1 branes the theory is globally invariant under the isometry group of $\mathbb{P}^{3|4}$ which is also the four dimensional superconformal group $\mathrm{SU}(2,2|4)$. The bosonic part, $\mathbb{P}^3$, of the supermanifold is identified with the twistor space of the four dimensional Minkowski space. The spectrum of physical states was analyzed [1] along the lines of [13] with the result that the physical states form the $\mathcal{N}=4$ $\mathrm{SU}(N)$ SYM multiplet. The classical equations of motion are, however, those of the self-dual YM theory and not those of the full $\mathcal{N}=4$ theory.

The conjecture of [1] is that the scattering amplitudes of the full $\mathcal{N}=4$ gauge theory are recovered by including D1 branes in this SFT. Counting of a particular U(1) charge violation and fermionic zero modes fixes the properties of the 1-brane contributing to an amplitude with fixed helicities and number of loops to the values listed in (2.7) and (2.8).

The introduction of an instantonic D1 brane leads to additional states localized on it. The 1-5 and 5-1 strings each contribute a single physical state α and β, respectively, in the fundamental representation of $\mathrm{SU}(N)$. It was argued in [1] that these fields should have fermionic statistics, opposite to the naive expectations. The 1-1 strings contribute a U(1) gauge field. The action for the α and β fields follows from the standard SFT action:

$$S=\int_{\mathcal{C}} dz\,\alpha(\bar{\partial}+\mathcal{A})\beta\,. \tag{3.1}$$

The integral is taken over the worldvolume of the D1-string, which wraps some holomorphic curve $\mathcal{C}$ sitting inside $\mathbb{P}^{3|4}$. Here $\mathcal{A}$ is the 5-5 string field and the coupling follows from $\mathrm{SU}(N)$ gauge invariance. The scattering amplitudes are then computed in terms of correlation functions of the currents $J(z)=\beta\alpha\,dz$ while treating $\mathcal{A}$ as a background field.

The introduction of a fixed D1 brane breaks most of the isometries of $\mathbb{P}^{3|4}$ and thus the resulting amplitudes cannot be invariant under four dimensional superconformal transformations. This apparent problem can be easily fixed by integrating over all possible configurations of the D1 brane, that is, over all possible choices of $\mathcal{C}$ in (3.1) with genus and degree determined by (2.7) and (2.8).

For genus zero, the moduli space of curves of degree d in $\mathbb{P}^{3|4}$ is most efficiently described in terms of degree d maps from $\mathbb{P}^1$ into $\mathbb{P}^{3|4}$. The embedding map as a function of the coordinate σ on $\mathbb{P}^1$ can be written in terms of $4(d+1)$ bosonic parameters and $4(d+1)$ fermionic parameters as

$$z^I=P^I(\sigma)=\sum_{k=0}^{d}a_k^I\sigma^k\,,\qquad \psi^A=G^A(\sigma)=\sum_{k=0}^{d}\beta_k^A\sigma^k\,, \tag{3.2}$$

where $z^I=(z^0,z^1,z^2,z^3)=(\lambda^1,\lambda^2,\mu^1,\mu^2)$ are the bosonic coordinates on $\mathbb{P}^{3|4}$, and ψ^A, $A=1,2,3,4$ are the fermionic coordinates. In this language, the integral over the moduli space of such curves becomes an integral over all a_k^I and β_k^A while dividing out by the GL(2) symmetry which acts in the obvious way on σ_i and (nonlinearly) on a and β.

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Combining these ingredients leads to the following master formula for the tree-level contribution to n-gluon scattering from instantons of degree d (relevant when there are $d+1$ negative helicity gluons).

$$B(\lambda,\mu,\psi)=\int\frac{d^{4d+4}a\; d^{4d+4}\beta\; d^n\sigma}{\mathrm{vol}(GL(2))}\mathcal{J}\prod_{i=1}^{n}\delta^3\left(\frac{z_i^I}{z_i^J}-\frac{P^I(\sigma_i)}{P^J(\sigma_i)}\right)\delta^4\left(\frac{\psi_i^A}{z_i^J}-\frac{G^A(\sigma_i)}{P^J(\sigma_i)}\right). \quad (3.3)$$

The 3-dimensional delta function is taken over $I\neq J$, where the choice of J is easily seen to be arbitrary. The final ingredient is the free fermion correlator

$$\mathcal{J}=\prod_{i=1}^{n}\frac{1}{\sigma_i-\sigma_{i+1}}. \quad (3.4)$$

Note that the coordinates z^I, ψ^A and σ_i, as well as the moduli a_k^I and β_k^A are all complex variables, so in writing the formula (3.3) we must specify an integration contour. In signature $--++$ it is natural to choose the naive contour where all variables lie on the real axis [1].

According to the proposal of [1], (3.3) gives a contribution to the tree-level YM amplitudes Fourier transformed to twistor space via (2.5). In [1] the possibility was considered that there might be additional contributions coming from separated instantons of lower degree. In the case we consider next, $n=5$ and $d=2$, we find agreement with YM theory without needing such contributions.

4. The B-model calculation for $n=5$, $d=2$

In this section we evaluate the B-model amplitude (3.3) for the case $n=5$, $d=2$, which is relevant to the scattering of 3 negative and 2 positive helicity gluons in YM theory. It is reasonably straightforward to evaluate the integral (3.3) directly in this case.[3] However, the quantity we are interested in comparing to a YM amplitude is not B, but its Fourier transform $\widetilde{B}$. It appears quite intractable to first calculate B and then take the Fourier transform, so we will proceed by taking the Fourier transform first, before evaluating the integral over moduli space.

For the next few steps we forget about the fermionic factor, restoring it at the end of the calculation. To simplify the already complicated notation, we define the rescaled variables $\lambda_i^2\to\lambda_i^2/\lambda_i^1$ and $\mu_i^{\dot a}\to\mu_i^{\dot a}/\lambda_i^1$. The dependence on λ^1 can be easily restored at the end, by the inverse transformation. Fourier transforming the original variable $\mu^{\dot a}\to\widetilde{\lambda}^{\dot a}$ then gives

$$\widetilde{B}(\lambda,\widetilde{\lambda})=\int\frac{d^{12}a\, d^5\sigma}{\mathrm{vol}(GL(2))}J_0\left[\prod_{i=1}^{5}\delta\left(\lambda_i^2-\frac{P^1(\sigma_i)}{P^0(\sigma_i)}\right)\right]\exp\left[i\sum_{i=1}^{5}\sum_{k=0}^{2}\frac{\epsilon_{\dot a\dot b}\widetilde{\lambda}_i^{\dot a}a_k^{\dot b}\sigma_i^k}{P^0(\sigma_i)}\right], \quad (4.1)$$

[3] Counting the bosonic delta functions reveals that the result will be proportional to two delta functions. A not so difficult calculation reveals that they are $\delta(K_{1234})\delta(K_{1235})$, where K is the object in [1] which expresses the constraint that four points lie on a plane $\mathbb{P}^2$ inside $\mathbb{P}^3$. Note that any five points which lie on a common plane automatically lie on a degree 2 curve. It is not clear how to directly Fourier transform (a complicated function times) $\delta(K_{1234})\delta(K_{1235})$ back to physical space.

where $\widetilde{\lambda}$ is related to the dual of μ by the dual rescaling $\widetilde{\lambda}_i^{\dot{a}} \to \widetilde{\lambda}_i^{\dot{a}} \lambda_i^1$ and we can absorb the associated factor of $\prod(\lambda_i^1)^2$ coming from the measure of the Fourier transform into

$$J_0 = \prod_{i=1}^{5} \frac{(\lambda_i^1)^2}{\sigma_i - \sigma_{i+1}} \,. \tag{4.2}$$

The first step is to fix the GL(2) symmetry by setting the variables a_0^0, σ_1, σ_2 and σ_3 to some arbitrary values at the cost of introducing the jacobian

$$J_1 = a_0^0 V_{123} \,, \qquad V_{123} \equiv (\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1) \,. \tag{4.3}$$

The integral over the six $a_k^{\dot{a}}$ moduli is trivial and gives

$$\widetilde{B} = \int d^2 a\, d^3 b\, d\sigma_4\, d\sigma_5 J_0 J_1 \left[\prod_{i=1}^{5} \delta\left(\lambda_i^2 - \frac{B_i}{A_i} \right) \right] \prod_{k=0}^{2} \delta^2 \left(\sum_{i=1}^{5} \frac{\widetilde{\lambda}_i^{\dot{a}} \sigma_i^k}{A_i} \right) . \tag{4.4}$$

Here we have parametrized the remaining bosonic moduli by a_k (with $a_0 = a_0^0$ unintegrated) and b_k, with

$$A_i = \sum_{k=0}^{2} a_k \sigma_i^k \,, \qquad B_i = \sum_{k=0}^{2} b_k \sigma_i^k \,. \tag{4.5}$$

The next step is to make use of the remarkable identity

$$\begin{aligned} \left[\prod_{i=1}^{5} \delta\left(\lambda_i^2 - \frac{B_i}{A_i} \right) \right] \prod_{k=0}^{2} \delta^2 \left(\sum_{i=1}^{5} \frac{\widetilde{\lambda}_i^{\dot{a}} \sigma_i^k}{A_i} \right) &= J_2 \delta^2 \left(\sum_{i=1}^{5} \widetilde{\lambda}_i^{\dot{a}} \right) \delta^2 \left(\sum_{i=1}^{5} \widetilde{\lambda}_i^{\dot{a}} \lambda_i^2 \right) \times \\ &\quad \times \left[\prod_{i=1}^{3} \delta\left(\lambda_i^2 - \frac{B_i}{A_i} \right) \right] \prod_{k=1}^{2} \times \\ &\quad \times \delta(S_4^k - \sigma_5 S_4^{k-1}) \delta(S_5^k - \sigma_4 S_5^{k-1}) \,, \end{aligned} \tag{4.6}$$

where

$$J_2 = A_4 A_5 [4,5]^3 \,, \qquad S_j^k = \sum_{i=1}^{3} \frac{[i,j]}{A_i} \sigma_i^k \,. \tag{4.7}$$

This identity has a number of useful consequences. Notably, the first two delta functions combine into overall delta function of momentum conservation $\delta^4(p) = \delta^4(\sum_{i=1}^5 \widetilde{\lambda}_i^{\dot{a}} \lambda_i^{\dot{a}})$ after restoring λ_i^1 dependence. Moreover, the σ_4 and σ_5 variables now appear linearly in the delta functions. These, as well as the three b moduli, can therefore be integrated out with ease. The latter give a jacobian of

$$J_3 = \frac{A_1 A_2 A_3}{V_{123}} \,, \tag{4.8}$$

and we are left with

$$\widetilde{B} = \delta^4(p) \int d^2 a\ J_0 J_1 J_2 J_3\ \delta(S_4^0 S_4^2 - (S_4^1)^2) \delta(S_5^0 S_5^2 - (S_5^1)^2) \,. \tag{4.9}$$

It is at this stage that the most remarkable feature of the identity (4.6) emerges. After substituting the definition (4.7) for S_j^k into (4.9), the remaining two delta functions turn out to be linear in the remaining moduli a_1, a_2! Integrating them out gives one final jacobian,

$$J_4 = \frac{A_1(A_2A_3)^2}{J_1 S_4^0 S_5^0([4,2][5,3](\sigma_2-\sigma_5)^2(\sigma_3-\sigma_4)^2 - [4,3][5,2](\sigma_2-\sigma_4)^2(\sigma_3-\sigma_5)^2)}\,. \tag{4.10}$$

Now we assemble all of the jacobians that have piled up along the way and plug in the values of the moduli and σ_4, σ_5 set by the various delta functions. In this way we obtain

$$\widetilde{B} = \delta^4(p)\left[\frac{[2,1][3,1][2,4][3,4][2,5][3,5]a_0^3(\sigma_2-\sigma_3)^6}{[4,1]^2[5,1]^2[3,2]^2(V_{123})^3}\right]^4 \prod_{i=1}^{5}\frac{(\lambda_i^1)^2}{[i,i+1]}\,. \tag{4.11}$$

There is significant ambiguity in writing this formula since a_0 is a completely free parameter — one could choose a_0 such that the whole quantity in brackets is 1, for example. In writing (4.11) we have chosen a_0 such that A_1, when evaluated on the solution of all the delta functions, is independent of σ_1, σ_2 and σ_3. This ambiguity will cancel against the fermionic determinant to be calculated next.

The final step is to evaluate the fermionic contribution to the amplitude,

$$F \equiv \int d^{12}\beta \prod_{i=1}^{5}\delta^4\left(\psi_i^A - \sum_{k=0}^{2}\frac{\beta_k^A\sigma_i^k}{A_i}\right). \tag{4.12}$$

To accomplish this we use a simple analogue of (4.6) which lets us pull out the super-momentum conservation constraint:

$$\prod_{i=1}^{5}\delta^4\left(\psi_i^A - \sum_{k=0}^{2}\frac{\beta_k^A\sigma_i^k}{A_i}\right) = \frac{1}{[4,5]^4}\delta^8\left(\sum_{i=1}^{5}\widetilde{\lambda}_i^{\dot a}\psi_i^A\right)\prod_{i=1}^{3}\delta^4\left(\psi_i^A - \sum_{k=0}^{2}\frac{\beta_k^A\sigma_i^k}{A_i}\right). \tag{4.13}$$

Inserting (4.13) into (4.12) and doing the β integrals immediately gives the fermionic determinant

$$F = \left[\frac{V_{123}}{A_1A_2A_3[4,5]}\right]^4 \delta^8\left(\sum_{i=1}^{5}\widetilde{\lambda}_i^{\dot a}\psi_i^A\right). \tag{4.14}$$

The quantities A_i are determined in terms of the $\widetilde{\lambda}_i^{\dot a}$ through the bosonic delta functions, all of which we have already demonstrated how to solve. Substituting the solutions gives the final expression

$$F = \delta^8\left(\sum_{i=1}^{5}\widetilde{\lambda}_i^{\dot a}\psi_i^A\right)\left[\frac{[4,1]^2[5,1]^2[3,2]^2(V_{123})^3}{[2,1][3,1][2,4][3,4][2,5][3,5]a_0^3(\sigma_2-\sigma_3)^6}\right]^4. \tag{4.15}$$

Combining (4.11) and (4.15) and restoring the λ_i^1 dependence by rescaling λ_i^2, $\widetilde{\lambda}_i^{\dot a}$ and ψ^A as explained in the beginning of this section yields the full B-model amplitude

$$\widetilde{B}(\lambda,\widetilde{\lambda},\psi) = \delta^4\left(\sum_{i=1}^{5}\widetilde{\lambda}_i^{\dot a}\lambda_i^a\right)\delta^8\left(\sum_{i=1}^{5}\widetilde{\lambda}_i^{\dot a}\psi_i^A\right)\frac{1}{[1,2][2,3][3,4][4,5][5,1]}\,, \tag{4.16}$$

in agreement with (2.9) after the necessary fermionic Fourier transform.

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